Exercise 1 (4 points)

A Givens rotation $G_{ij}(\theta)$ is a rotation through an angle $\theta$ in the plane spanned by the unit vectors $e_i$ and $e_j$, $i < j$. The corresponding matrix

$$G_{ij}(\theta) = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & c & s & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & c & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -s & c & \cdots & \cdots & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{m \times m}$$

with $c = \cos \theta$ and $s = \sin \theta$ is a rank two modification of the unit matrix $I_m$.

(a) Show that $G_{ij}(\theta)$ is unitary.

(b) Give a formula for real numbers $c$ and $s$ with $c^2 + s^2 = 1$ such that for given $\alpha, \beta \in \mathbb{R}$ you get

$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad \text{where } \sigma \in \mathbb{R}.$$ 

(c) Give an algorithm for the QR-factorization of an upper Hessenberg matrix $H_{j+1,j} \in \mathbb{R}^{j+1 \times j}$ using Givens rotations.

Exercise 2 (3 points)

Show that the computational cost of the Householder QR-factorization of an $m \times n$ matrix is

$$2mn^2 - \frac{2}{3}n^3 \quad \text{flops.}$$
Exercise 3 (6 points)
Take \( m = 50, n = 12 \). Using MATLAB’s \texttt{linspace}, define \( t \) to be the \( m \)-vector corresponding to linearly spaced grid points from 0 to 1. Using MATLAB’s \texttt{vander} and \texttt{flipud}, define \( A \) to be the \( m \times n \) matrix associated with least squares fitting on this grid by a polynomial of degree \( n - 1 \). Take \( b \) to be the function \( \cos(4t) \) evaluated on the grid. Now, calculate and print (to sixteen-digit precision) the least squares coefficient vector \( x \) by six methods:

- Formation and solution of the normal equations, using MATLAB’s \texttt{\backslash},
- QR factorization computed by \texttt{mgs} (modified Gram-Schmidt, sheet 4),
- QR factorization computed by \texttt{house_qr} (Householder triangularization, sheet 7),
- QR factorization computed by MATLAB’s \texttt{qr} (which also uses Householder triangularization),
- \( x = A\backslash b \) in MATLAB (also based on QR factorization),
- SVD, using MATLAB’s \texttt{svd}.

The calculations above will produce six lists of twelve coefficients. In each list, mark the digits that appear to be wrong (affected by rounding error). Comment on what differences you observe. Do the normal equations exhibit instability?

Exercise 4 (2 points)
Consider the polynomial \( p(x) = (x - 2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512 \).

(a) Plot \( p(x) \) for \( x = 1.920, 1.921, 1.922, \ldots, 2.080 \), evaluating \( p \) via its coefficients 1, \(-18, 144, \ldots\)

(b) Produce the same plot again, now evaluating \( p \) via the expression \( (x - 2)^9 \).